Problem 16)

a) $$J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}\,d\theta=\frac{1}{2\pi}\left\{\int_{-\pi}^{-\pi/2} e^{ix\sin\theta}\,d\theta+\int_{-\pi/2}^{\pi/2} e^{ix\sin\theta}\,d\theta+\int_{\pi/2}^{\pi} e^{ix\sin\theta}\,d\theta\right\}.$$

Change of Variable: $\sin\theta=2\pi s \Rightarrow \cos\theta\,d\theta=2\pi ds$,

$$d\theta=\frac{2\pi ds}{\cos\theta}=\frac{2\pi ds}{\pm\sqrt{1-\sin^2\theta}}=\frac{2\pi ds}{\pm\sqrt{1-4\pi^2 s^2}}$$

+ sign when $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
− sign when $-\pi\le\theta<-\frac{\pi}{2}$ and $\frac{\pi}{2}\le\theta\le\pi$.

Therefore, $$J_0(x)=\frac{1}{2\pi}\left\{\int_0^{-\frac{1}{2\pi}}\frac{2\pi}{-\sqrt{1-4\pi^2s^2}}e^{i2\pi sx}ds+\int_{-\frac{1}{2\pi}}^{1/2\pi}\frac{2\pi}{\sqrt{1-4\pi^2s^2}}e^{i2\pi sx}ds+\int_{\frac{1}{2\pi}}^{0}\frac{2\pi}{-\sqrt{1-4\pi^2s^2}}e^{i2\pi sx}ds\right\}$$

$$\Rightarrow J_0(x)=2\int_{-1/2\pi}^{1/2\pi}\frac{e^{i2\pi sx}}{\sqrt{1-4\pi^2s^2}}ds=2\int_{-\infty}^{\infty}\frac{\mathrm{Rect}(\pi s)}{\sqrt{1-4\pi^2s^2}}e^{i2\pi sx}ds$$

The above inverse F.T. integral shows that $\mathcal{F}\{J_0(x)\}=\frac{2\,\mathrm{Rect}(\pi s)}{\sqrt{1-4\pi^2s^2}}$.

Using the scaling theorem, we'll have:

$$\mathcal{F}\{J_0(2\pi x)\}=\frac{1}{2\pi}\mathcal{F}\{J_0(x)\}\Big|_{s/2\pi}=\frac{\mathrm{Rect}(s/2)}{\pi\sqrt{1-s^2}}.$$

Second method: $$\mathcal{F}\{J_0(2\pi x)\}=\int_{-\infty}^{\infty}J_0(2\pi x)e^{-i2\pi sx}dx=$$

$$\int_{-\infty}^{0}J_0(2\pi sx)e^{-i2\pi sx}dx+\int_0^{\infty}J_0(2\pi x)e^{-i2\pi sx}dx=2\int_0^{\infty}J_0(2\pi x)\cos(2\pi sx)\,dx$$

The above integral is given in Gradshteyn & Ryzhik (6.671-2) as follows:

$$\int_0^\infty J_0(\alpha x)\cos\beta x\,dx = \begin{cases} \dfrac{1}{\sqrt{\alpha^2-\beta^2}} ; & \beta<\alpha, \\ 0 ; & \beta>\alpha. \end{cases}$$

Therefore, $$\mathcal{F}\{J_0(2\pi x)\} = \begin{cases} \dfrac{2}{\sqrt{4\pi^2-4\pi^2 s^2}} ; & |s|<1 \\ 0; & |s|>1 \end{cases} = \frac{\mathrm{Rect}(s/2)}{\pi\sqrt{1-s^2}} .$$

Third method: The Bessel equation for $\nu=0$ yields the following relation among $J_0(x)$, $J_0'(x)$, and $J_0''(x)$:

$$x J_0''(x) + J_0'(x) + x J_0(x) = 0$$

Denoting the Fourier transform of $J_0(x)$ by $\underset{\sim}{J_0}(s)$, and remembering that $\mathcal{F}\{J_0'(x)\} = i2\pi s\,\underset{\sim}{J_0}(s)$ and $\mathcal{F}\{J_0''(x)\} = (i2\pi s)^2\underset{\sim}{J_0}(s)$, we write the above equation as follows:

$$x\int_{-\infty}^{\infty}(-4\pi^2 s^2)\underset{\sim}{J_0}(s)e^{+i2\pi s x}ds + \int_{-\infty}^{\infty} i2\pi s\,\underset{\sim}{J_0}(s)e^{+i2\pi s x}ds + x\int_{-\infty}^{\infty}\underset{\sim}{J_0}(s)e^{+i2\pi s x}ds = 0$$

Next, we Fourier-transform both sides of the equation, denote the frequency by s' (to distinguish it from the dummy variables), and change the order of integration to arrive at:

$$-4\pi^2\int_{-\infty}^{\infty}s^2\underset{\sim}{J_0}(s)\left[\int_{-\infty}^{\infty}x e^{i2\pi(s-s')x}dx\right]ds + i2\pi\int_{-\infty}^{\infty}s\,\underset{\sim}{J_0}(s)\left[\int_{-\infty}^{\infty}e^{i2\pi(s-s')x}dx\right]ds$$

$$+\int_{-\infty}^{\infty}\underset{\sim}{J_0}(s)\left[\int_{-\infty}^{\infty}x e^{i2\pi(s-s')x}dx\right]ds = 0$$

$$\Rightarrow -4\pi^2\int_{-\infty}^{\infty} s^2 \tilde{J}_0(s)\,\frac{\delta'(s-s')}{i2\pi}\,ds + i2\pi\int_{-\infty}^{\infty} s\,\tilde{J}_0(s)\,\delta(s-s')\,ds + \int_{-\infty}^{\infty}\tilde{J}_0(s)\,\frac{\delta'(s-s')}{i2\pi}\,ds = 0$$

$$\Rightarrow -i2\pi\,\frac{d}{ds}\left[s^2\tilde{J}_0(s)\right]_{s'} + i2\pi s'\tilde{J}_0(s') + \frac{i}{2\pi}\frac{d}{ds}\left[\tilde{J}_0(s)\right]_{s'} = 0$$ ← sifting prop. of $\delta(s)$ and $\delta'(s)$

$$\Rightarrow 2s'\tilde{J}_0(s') + s'^2\tilde{J}_0'(s') - s'\tilde{J}_0(s') - \frac{1}{4\pi^2}\tilde{J}_0'(s') = 0$$

At this point there is no need to use s', and we revert to using s as the frequency variable in the Fourier domain. Rearranging the terms in the above equation, which is the F.T. of the Bessel equation for $J_0(x)$, we'll have:

$$s\tilde{J}_0(s) + \left(s^2 - \frac{1}{4\pi^2}\right)\tilde{J}_0'(s) = 0 \Rightarrow \frac{\tilde{J}_0'(s)}{\tilde{J}_0(s)} = \frac{4\pi^2 s}{1-4\pi^2 s^2} \Rightarrow$$

$$\frac{d}{ds}\ln\tilde{J}_0(s) = -\frac{1}{2}\frac{d}{ds}\ln(1-4\pi^2 s^2) \Rightarrow \ln\tilde{J}_0(s) = \ln\frac{1}{\sqrt{1-4\pi^2 s^2}} + c'$$

$$\Rightarrow \tilde{J}_0(s) = \frac{c}{\sqrt{1-4\pi^2 s^2}}.$$

Note that this method can't determine $\tilde{J}_0(s)$ where $\tilde{J}_0(s) = 0$. The correct answer obtained in part (a) yields $\tilde{J}_0(s) = \dfrac{2\,\mathrm{Rect}(\pi s)}{\sqrt{1-4\pi^2 s^2}}$. The present method yields the correct answer only when $|s| < \frac{1}{2\pi}$, but we have no way of knowing it.

b) $$\mathcal{F}\{J_1(x)\} = -\mathcal{F}\{J_0'(x)\} = -i2\pi s\,\tilde{J}_0(s) = -\frac{i4\pi s\,\mathrm{Rect}(\pi s)}{\sqrt{1-4\pi^2 s^2}}$$

$$\Rightarrow \mathcal{F}\{J_1(2\pi x)\} = \frac{1}{2\pi}\tilde{J}_1\left(\frac{s}{2\pi}\right) = -\frac{i s\,\mathrm{Rect}(s/2)}{\pi\sqrt{1-s^2}}$$

Scaling theorem

We also know that $\mathcal{F}\{\text{Sign}(x)\} = -\frac{i}{\pi s} \Rightarrow \mathcal{F}\{\frac{1}{x}\} = -i\pi\,\text{Sign}(s)$.
Using the Convolution theorem we write:

$$\mathcal{F}\left\{\frac{J_1(2\pi x)}{2x}\right\} = \frac{1}{2}\mathcal{F}\{J_1(2\pi x)\} * \mathcal{F}\{\tfrac{1}{x}\} =$$

$$\frac{1}{2}\left[\frac{-is\,\text{Rect}(s/2)}{\pi\sqrt{1-s^2}}\right] * \left[-i\pi\,\text{Sign}(s)\right] = -\frac{1}{2}\,\frac{s\,\text{Rect}(s/2)}{\sqrt{1-s^2}} * \text{Sign}(s)$$

Since $\frac{s\,\text{Rect}(s/2)}{\sqrt{1-s^2}}$ is an odd function of s, its convolution with Sign(s) will be zero if $|s|$ happens to be greater than 1. For $|s|<1$ we'll have

$$-\frac{s\,\text{Rect}(s/2)}{2\sqrt{1-s^2}} * \text{Sign}(s) = -\int_{-1}^{s}\frac{s'}{2\sqrt{1-s'^2}}\,ds' + \int_{s}^{+1}\frac{s'}{2\sqrt{1-s'^2}}\,ds'$$

$$= \frac{1}{2}\sqrt{1-s'^2}\,\Big|_{-1}^{s} - \frac{1}{2}\sqrt{1-s'^2}\,\Big|_{s}^{+1} = \sqrt{1-s^2}.$$

Therefore, $\mathcal{F}\left\{\frac{J_1(2\pi x)}{2x}\right\} = \sqrt{1-s^2}\,\text{Rect}(s/2).$

c) $\mathcal{F}\{\text{Sign}(x)\,J_0(2\pi x)\} = \int_{-\infty}^{\infty}\text{Sign}(x)\,J_0(2\pi x)\,e^{-i2\pi s x}\,dx =$

$$-\int_{-\infty}^{0}J_0(2\pi x)\,e^{-i2\pi s x}\,dx + \int_{0}^{\infty}J_0(2\pi x)\,e^{-i2\pi s x}\,dx = \int_{0}^{\infty}J_0(2\pi x)\left(e^{-i2\pi s x} - e^{+i2\pi s x}\right)dx$$

$$= -2i\int_{0}^{\infty}J_0(2\pi x)\sin(2\pi s x)\,dx$$

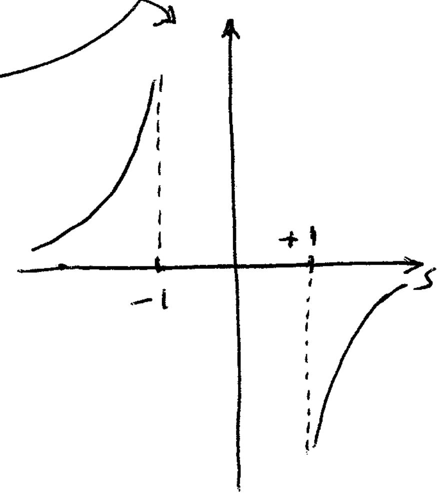
The above integral is given in Gradshteyn + Ryzhik (6.671-1) as follows:

$$\int_0^\infty J_0(\alpha x)\sin(\beta x)\,dx = \begin{cases} 0; & \beta<\alpha \\ \dfrac{1}{\sqrt{\beta^2-\alpha^2}}; & \beta>\alpha \end{cases}.$$

Note that the above integral will have to be multiplied by -1 if β were negative. Consequently:

$$\mathcal{F}\{\text{Sign}(x)J_0(2\pi x)\} = -2i\,\text{Sign}(s)\begin{cases} 0; & |s|<1 \\ \dfrac{1}{\sqrt{4\pi^2 s^2-4\pi^2}}; & |s|>1 \end{cases} \Rightarrow$$

$$= i\,\text{Sign}(s)\,\frac{\text{Rect}(s/2)-1}{\pi\sqrt{s^2-1}}.$$



Alternative method:

$$\mathcal{F}\{\text{Sign}(x)J_0(2\pi x)\} = -\frac{i}{\pi s} * \frac{\text{Rect}(s/2)}{\pi\sqrt{1-s^2}}$$

$$= -\frac{i}{\pi^2}\int_{-1}^{+1}\frac{1}{\sqrt{1-s'^2}}\cdot\frac{1}{s-s'}\,ds'$$

Change of variable: $s'=\cos\theta \Rightarrow ds' = -\sin\theta\,d\theta = -\sqrt{1-s'^2}\,d\theta$. Therefore,

$$\mathcal{F}\{\text{Sign}(x)J_0(2\pi x)\} = +\frac{i}{\pi^2}\int_\pi^0 \frac{1}{\cancel{\sqrt{1-s'^2}}}\,\frac{1}{s-\cos\theta}\,\cancel{\sqrt{1-s'^2}}\,d\theta = -\frac{i}{2\pi^2}\int_0^{2\pi}\frac{d\theta}{s-\cos\theta}.$$

We have seen in Problem (29) that $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ provided that $a>|b|$.

Therefore, $\int_0^{2\pi}\frac{d\theta}{s-\cos\theta} = \frac{2\pi}{\sqrt{s^2-1}}$ if $s>1$ and $-\frac{2\pi}{\sqrt{s^2-1}}$ if $s<-1$. These are consistent with the result obtained with the first method. We still need to show that $\int_0^{2\pi} \frac{d\theta}{S-\cos\theta} = 0$ when $|S|<1$. (This integral is divergent, of course, and it is its Principal Value that must be evaluated.) We write:

$$\int_0^{2\pi} \frac{d\theta}{S-\cos\theta} = \int_0^{2\pi} \frac{d\theta}{S-\frac{1}{2}(e^{i\theta}+e^{-i\theta})} = \int_0^{2\pi} \frac{-2e^{i\theta}\,d\theta}{e^{2i\theta}-2Se^{i\theta}+1} = -2\int_0^{2\pi} \frac{e^{i\theta}\,d\theta}{(e^{i\theta}-e^{i\theta_0})(e^{i\theta}-e^{-i\theta_0})}$$

The θ_0 appearing in the denominator is chosen such that $S=\cos\theta_0$. Since $-1<S<+1$, this choice of θ_0 is certainly feasible. With $0<\theta_0<\pi$, we also have $0<\sin\theta_0<1$.

$$\frac{1}{(e^{i\theta}-e^{i\theta_0})(e^{i\theta}-e^{-i\theta_0})} = \frac{A}{e^{i\theta}-e^{i\theta_0}} + \frac{B}{e^{i\theta}-e^{-i\theta_0}} \Rightarrow A=-B=-\frac{i}{2\sin\theta_0}$$

Therefore, $\int_0^{2\pi} \frac{d\theta}{S-\cos\theta} = \frac{i}{\sin\theta_0}\left\{\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}-e^{i\theta_0}}\,d\theta - \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}-e^{-i\theta_0}}\,d\theta\right\}$

$$= \frac{i}{\sin\theta_0}\left\{\int_0^{2\pi} \frac{d\theta}{1-e^{-i(\theta-\theta_0)}} - \int_0^{2\pi} \frac{d\theta}{1-e^{-i(\theta+\theta_0)}}\right\}$$

Both integrands are periodic, each having a period of 2π. Since the integrals are taken over a full period, a shift of the variable θ by a constant value of $+\theta_0$ or $-\theta_0$ does not change the value of the integral. Therefore, the two integrals, being identical, cancel each other, yielding a net value of zero for $\int_0^{2\pi} \frac{d\theta}{S-\cos\theta}$.